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INTERACTION OF A PIEZOELECTRIC TRANSDUCER WITH AN ELASTIC HALF-SPACE

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Abstract—We study the interaction between an ultrasonic transducer, typical of those presently used in Nondestructive Testing (NDT) practice, and a test medium, through a viscous couplant. The transducing element is a circular piezoelectric cylinder of class C_{6v}, polarized in the thickness direction with electroded faces and insulated lateral surface. The electrodes are connected by an electric circuit, and the transducer is considered to be either sending or receiving, depending on whether or not a course of specified voltage is included in the circuit. The piezoelectric disc is modeled as a first order linear plate of finite extent. The test medium is idealized to be an elastic half-space.

We present the steady vibration problem of the transducer/test medium configuration forced by the voltage source in the transducer circuit and an incident wave in the test medium. This problem is reduced to coupled integral equations for the torsionless axisymmetric components of the interface tractions on the piezoelectric disc's faces. Various approximations of the problem are also considered which allow us to obtain the transducer output without solving these integral equations.

1. INTRODUCTION

This paper is concerned with the interaction of an electro-mechanical transducer, either sending or receiving, with an elastic half-space. It is motivated by a desire to increase the understanding of the engineering practice of Nondestructive Testing (NDT). In NDT work a receiving transducer, consisting of a transducing element that is mounted in a complicated casing and coupled to a test medium surface, receives mechanical wave disturbances generated either by a sending transducer or an event such as a spontaneous fracture. The sending transducer may be a device similar to the receiving transducer, except that a driving voltage source is included in its electric circuit. From the electrical output of the receiving transducer, i.e. the measured voltage drop across the transducing element or the current in the circuit, the tester wishes to determine the location, geometry and orientation of flaws present in the test medium.

The interaction between the transducer unit and the test medium is complex. The transduction process involves many mechanisms which are not yet well understood, such as the transducer/test medium coupling, and the transducer unit typical of those commercially available for NDT applications is composed of various materials and shapes that often do not lend themselves to a simple analysis.

As described in Sachse and Hsu[1], the usual treatment of the interaction involves a reduction of the transduction process to a one dimensional linear model by adopting three simplifying assumptions. The transducer bottom surface is coupled in some manner to a portion ∂B of the surface of the test medium. The transducing element has across it a voltage $V(t)$ and is connected electrically to a circuit carrying current $I(t)$ (see Fig. 1). In [1] the transduction process is represented symbolically as

$$\frac{V(t)}{I(t)} \Leftrightarrow \frac{\mathbf{T}(\mathbf{x}, t)}{\mathbf{u}(\mathbf{x}, t)}, \tag{1.1}$$

where $\mathbf{T}(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}, t)$ are the traction and displacement, respectively, on the contact portion ∂B of the test medium surface. In the usual treatment, the first simplifying assumption is that the transducer element deformation occurs in a single mode, i.e. the

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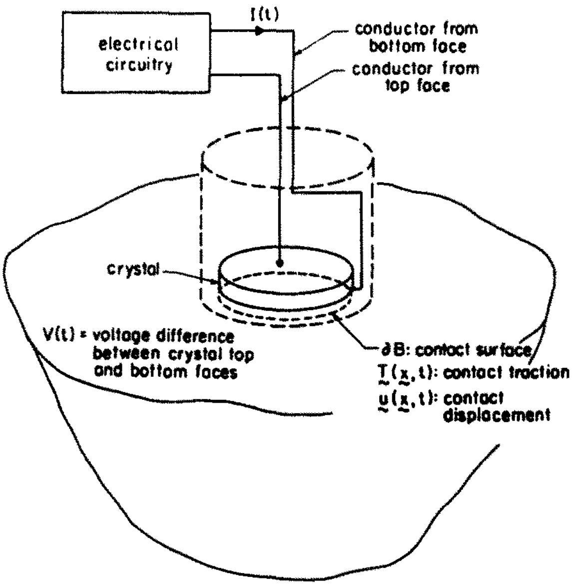

Fig. 1. Transducer unit and test medium.

wave modes are uncoupled and only a single scalar component of each of the traction and displacement are related to the voltage and current. For a compressional mode transducer, as will be studied in this paper, this assumption reduces eqn (1.1) to

$$\frac{V(t)}{I(t)} \Leftrightarrow \frac{T_z(\mathbf{x}, t)}{u_z(\mathbf{x}, t)}, \tag{1.2}$$

where the z direction is normal to the test medium surface ∂B. The second assumption is that ∂B is small and/or the traction and displacement fields are uniform over ∂B, so that eqn (1.2) reduces to

$$\frac{V(t)}{I(t)} \Leftrightarrow \frac{T_z(t)}{u_z(t)}. \tag{1.3}$$

These first two assumptions reduce the transduction process to a one dimensional model. The third assumption usually made is that the relationships between the scalar quantities $V(t)$, $I(t)$, $T_z(t)$ and $u_z(t)$ are linear so that they are related by a transduction matrix according to

$$\begin{bmatrix} V(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} T_z(t) \\ u_z(t) \end{bmatrix}. \tag{1.4}$$

For purposes of transducer design, one dimensional analyses can be used to express the transduction matrix in terms of quantities pertaining to the various elements of the transducer unit and couplant (to characterize existing transducers most often the device is assumed to be linear and one dimensional and the transduction matrix is obtained experimentally; see [1]). These analyses fall into two groups, the first of which treats the transducer as an infinite plate executing a specified motion in a particular thickness mode. An example of this group is found in Meeker[2], where it is assumed that the transducing element is an infinite piezoelectric plate governed by the linear three dimensional theory (Tiersten[3]), with boundary conditions and material symmetries that allow a specified thickness mode of vibration. Relations between the boundary tractions and displacements and transducer voltage and current are explicitly expressed in terms of the piezoelectric constants. If the plate is attached to a backing or coupled to a structure, however, impedances must be postulated, which can only be determined experimentally.

The second group of analyses is based on the equivalent circuit approach, as in Mason[4]. Here the transducer is assumed to be a finite plate executing a specified one dimensional motion (no attempt is made to show that this motion can be supported in the three dimensional continuum as is done in the previous group of analyses; it is understood that the other coupled modes are neglected). The resulting equations involving face resultant forces and displacements, current and voltage are recognized as being identical to those resulting from the analysis of an "equivalent" circuit, in which the scalar force and displacement components, functions only of time, are treated as voltage and current, respectively. As more mechanical elements, e.g. backings and couplants, are added to the transducer, additional impedances are inserted in the equivalent circuit (see Kossof[5] and Sittig[6]). It is shown by Meeker[2] that the results of the equivalent circuit approach can be the same as those given by the infinite plate approach.

The above-mentioned one dimensional analyses ignore the effect on the transducer voltage and current of crystal deformation dependent on the in plane coordinates, called contour modes. In the first group contour modes cannot exist since the plate is assumed to be infinite, and in the second group such modes are neglected. In the works of Bugdayci and Bogy[8], Bogy and Bechtel[9] and Bogy and Mui[10] experimental results for a piezoelectric disc with traction free edge and specified face tractions show a substantial, and in some cases dominant, dependence of the voltage across the faces on the radial modes. When the linear first order plate theory derived in Bugdayci and Bogy[7] is used to analyze this problem, this dependence is predicted. For the piezoelectric disc coupled to an elastic half-space, this dependence on the radial modes is also found to be strong, both experimentally (Bogy and Su[11]) and analytically (Bechtel[12]).

Hence, in order for a real transducer to respond as a one dimensional device, the piezoelectric disc must be modified considerably, e.g. by the addition of front plates and backing materials to effect heavy damping and impedance matching. The resulting typical commercial transducer unit is then without many of the response characteristics that make the bare piezoelectric disc undesirable as a transducer for NDT work. However, its design is much more complicated than the simple crystals originally analysed by Meeker[2] or Mason[4], and to perform an accurate analysis of it is difficult, as will be evidenced in Section 2 of this paper.

We consider a transducer unit typical of the backed commercial devices currently available for NDT applications. Its transducing element is a piezoelectric cylinder which is an element in an electric circuit. For the case of a sending transducer we include a specified voltage source in the circuit, rather than merely specifying the voltage difference between the crystal electrodes. The transducer bottom surface contacts the surface of the test medium through a viscous couplant.

In the analysis of the transducer/test medium interaction we make only the third of the three previously listed simplifying assumptions on the transduction process, that it is linear. The piezoelectric disc is modeled as a first order linear plate of finite extent, so as to include the effects of the contour modes, using the theory derived in [7] and developed in [9]. We include the effects of the backing and constraining casing. However, we do not postulate, for instance, that the backing is impedance matched with the disc, or that the transducer/test medium assembly is "tuned" so that the crystal motion is entirely or primarily in a thickness dilatation mode. Results of the analysis would be to identify the backing material parameters for which an impedance match will occur, and to determine if the above-mentioned tuning is possible. A particular frequency bandwidth of operation is not specified *a priori*, except that the piezoelectric plate theory should be applicable.

In Section 2 the steady vibration problem of the transducer unit excited by a harmonic voltage source, harmonic forces on the casing outer surface and an incident surface wave on the test medium is formulated and reduced to a set of integral equations for components of the interface tractions on the piezoelectric crystal top and bottom surfaces. These integral equations are not solved in this paper, but in Section 3 we present approximations to the problem formulations which allow us to obtain parts of the complete solution, e.g. the transducer current and voltage, without solving the integral equations explicitly for the interface traction components.

In our analyses we are primarily concerned with the transduction process between the voltage across the piezoelectric disc (and the current in the transducer circuit) and the transducer/test medium configuration deformations and tractions. We will focus our attention to those parts of the problems that involve the transducer voltage and current, and we will deduce the following important result: For an axisymmetric transducer unit of the type considered here, the voltage $V(t)$ and current $I(t)$ couple only with the torsionless axisymmetric parts of the transducer/test medium deformations and tractions. This is to be contrasted with the first two assumptions of the conventional treatment, i.e. that (1) $V(t)$ and $I(t)$ couple only with the normal components of the interface tractions and displacement (for the thickness dilatation mode transducers studied here), and (2) these components are functions only of time and are independent of position (see eqns (1.2) and (1.3)).

Section 4 presents a summary and some conclusions.

2. TIME HARMONIC PROBLEM

We consider the steady vibration problem of a transducer, typical of those presently used in NDT practice, forced by a voltage source in the transducer circuit, tractions on its casing outer surface, and an incident straight crested surface wave on the test medium to which it is coupled. All of these forcing functions are assumed here to be harmonic in time with frequency ω. The basic structure of a typical commercial ultrasonic transducer unit is shown in Fig. 2 (see [13]). The transducing element in the transducer is assumed to be a piezoelectric circular cylinder of class $C_{6\omega}$, polarized in the thickness direction, with electroded top and bottom surfaces. Its lateral surface is electrically insulated. The bottom electroded surface is covered by a wear resistant plate, and to the top surface is attached a backing material, most commonly tungsten powder in epoxy. These elements are enclosed in a protective metal and plastic casing.

The frontface is mechanically coupled to the test medium surface by a viscous fluid, as is the usual practice (see [14]). This surface is usually planar in the coupling region. The top and bottom electroded surfaces of the piezoelectric disc are connected electrically through a circuit.

When the unit is employed as a receiving transducer, the incident mechanical signal, travelling through the test medium, is transmitted through the viscous couplant and excites a voltage across the piezoelectric crystal and a current in the transducer circuit. When used as a sending transducer, the voltage source in the transducer circuit causes the piezoelectric crystal to mechanically excite the test structure through the couplant.

The problem described above is very complex, due primarily to the complicated structure of the transducer unit and the nature of the transducer/test medium coupling. Here we model this problem so that the formulation is reasonably accurate without being so involved as to prevent any meaningful progress toward a solution. Hence, we must decide which features of the transducer/test medium configuration warrant a precise modeling, and which

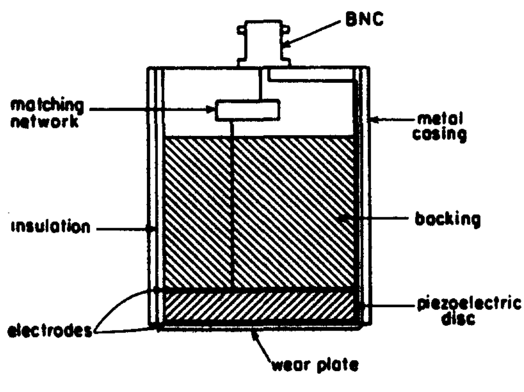

Fig. 2. Basic structure of an ultrasonic transducer based on a piezoelectric disc (from Bond *et al.*[15]).

may be neglected or modeled in a simple way. A more complete discussion of this modeling can be found in [12], on which this paper is based.

The piezoelectric crystal and transducer circuit are the same as discussed in Bogy and Bechtel[9], and they are modeled by the first-order plate theory and circuit equation developed in Bugdayci and Bogy[7] and [9]. It should be emphasized that the transducer circuit is the electrical circuit in which the piezoelectric disc is an element, and not an "equivalent" circuit as discussed in Section 1. The test medium is assumed to be a linear elastic isotropic homogeneous half-space. The presence of the wear plate on the frontface and any effects it has on the crystal/elastic half-space coupling are ignored. This coupling is modeled as in Angel and Bogy[15], i.e. the couplant which bonds the crystal bottom surface to the elastic half-space surface is assumed to be of negligible thickness and its inertial effects are ignored; in the contact region the displacements of the crystal bottom surface and half-space surface in the directional normal to the interface coincide, while the velocity difference in a direction along the interface is proportional by a factor κ to the traction component in that direction. The part of the half-space surface not coupled to the crystal is assumed traction free. The gravity body forces in both the crystal and the half-space equations of motion are neglected.

The insulation is considered as part of the casing, and the casing and backing are assumed to be symmetric about the axis of the crystal cylinder. We further assume that the casing is rigid.

It has been shown in Bogy and Su[11] that the backing bonded to the crystal top surface has the greatest effect on the response, e.g. altering the front wear plate or crystal lateral surface mechanical conditions produces less change in the response than changing the crystal backing conditions. The presence of the backing reduces the radial and thickness mode ringing that was shown in[8–11] to dominate the response of the unbacked crystal. This is a result of the damping properties of the backing material and the impedance match between the crystal and the backing. This impedance match results in a low reflection coefficient of waves in the crystal that are incident on the crystal/backing interface. Therefore, in order to accurately model a typical transducer unit, which exhibits little radial and thickness mode ringing, it is essential for the effect of the backing to be adequately represented. In [12] we consider two models for the backing, a viscoelastic continuum and a visco-elastic foundation. In this paper we present the analysis only for the latter.

As in [9] the piezoelectric crystal motion is referred to a cylindrical polar coordinate system (r, θ, z) with origin at the center of the crystal and $\mathbf{e}_z$ along the axis of symmetry. The casing motion is also referred to this coordinate system. The elastic half-space motion is referred to a primed system (r', θ', z') given by (see Fig. 3)

$$r' = r, \theta' = -\theta, z' = -z - b \tag{2.1}$$

where b is half of the crystal thickness.

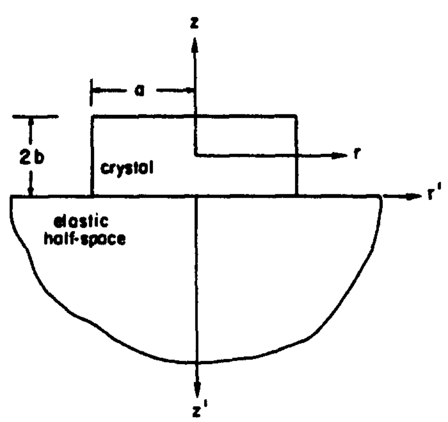

Fig. 3. Coordinate system placement.

According to the first order plate theory of [7, 9], we assume the displacement and potential in the crystal to be of the form

$$u_j(r,\theta,z,t) = u_j^{(0)}(r,\theta,t) + u_j^{(1)}(r,\theta,t)\cos\frac{\pi}{2}\left(1-\frac{z}{b}\right),\ j = r,\theta,z,$$

$$\phi(r,\theta,z,t) = \tfrac{1}{2}[\phi^+(t)+\phi^-(t)] + \frac{z}{b}B(t) + \phi^{(1)}(r,\theta,t)\sin\frac{\pi}{2}\left(1-\frac{z}{b}\right), \tag{2.2}$$

where $\phi^+(t)$ and $\phi^-(t)$ are the electrical potentials at the top and bottom surfaces, respectively, of the crystal and $B(t)$ is half of the voltage drop $V(t)$ across the crystal, i.e.

$$B(t) = [\phi^+(t) - \phi^-(t)]/2 = V(t)/2. \tag{2.3}$$

For the crystal and elastic half-space we define theta averaged functions by

$$\mathring{f}(r,t) = (2\pi)^{-1}\int_{-\pi}^{\pi} f(r,\theta,t)\,\mathrm{d}\theta,$$

$$\mathring{f}'(r',z',t) = (2\pi)^{-1}\int_{-\pi}^{\pi} f'(r',\theta',z',t)\,\mathrm{d}\theta', \tag{2.4}$$

respectively.

In [12] it is demonstrated that the transduction process of eqn (1.1) couples the transducer voltage $2B(t)$ and transducer current $I(t)$ with only the torsionless axisymmetric parts of the transducer unit and structure deformations, not with their theta dependent or axisymmetric torsion parts. The result differs from the common simplifying assumptions 1 and 2 as discussed in the Introduction (eqns (1.1) and (1.2)), and is due to the presence of the face electrodes on the piezoelectric crystal and the axisymmetry of our models for the transducer unit, couplant and test medium. Since we are primarily interested here in obtaining output $2B(t)$ and $I(t)$ of the trandsucer, we consider only the torsionless axisymmetric part of the problem for conditions of steady vibration.

For a general theta-averaged function $\mathring{f}(r,t)$ defined on the crystal plate we specify the time dependence by

$$\mathring{f}(r,t) = \mathrm{Re}\,\{\mathring{\bar{f}}(r)\,\mathrm{e}^{-i\omega t}\}. \tag{2.5}$$

Likewise, on the elastic half-space we write

$$\mathring{f}'(r',z',t) = \mathrm{Re}\,\{\mathring{\bar{f}}'(r',z')\,\mathrm{e}^{-i\omega t}\}. \tag{2.6}$$

For functions of time only, e.g. $B(t)$, we write

$$f(t) = \mathrm{Re}\,\{\bar{f}\,\mathrm{e}^{-i\omega t}\}. \tag{2.7}$$

Let the elastic half-space displacement be composed of incident and scattered parts, where the incident part is prescribed to be a straight crested surface wave travelling in the x_1' direction, and the scattered part vanishes at points remote from the crystal/elastic half-space interface. Then, using superscripts (I) and (S) to denote "incident" and "scattered", we have

$$\mathbf{u}' = \mathbf{u}^{(I)} + \mathbf{u}^{(S)},\ \lim_{|\mathbf{x}'|\to\infty} \mathbf{u}^{(S)}(\mathbf{x}',t) = 0,$$

$$u_1^{(I)}(x_i',t) = \mathrm{Re}\,\{\check{u}_1^{(I)}(x_2')\,\mathrm{e}^{-i\omega(t-s_R x_1')}\},\ u_3^{(I)}(x_i',t) = 0,$$

$$u_2^{(I)}(x_i',t) = \mathrm{Re}\,\{\check{u}_2^{(I)}(x_2')\,\mathrm{e}^{-i\omega(t-s_R x_1')}\}, \tag{2.8}$$

$$\breve{u}_1^{(I)}(x_2') = u_0\eta_R^{-2}[2\,e^{-\omega s_R\alpha_1 x_2'} - (2-\eta_R^2)\,e^{-\omega s_R\alpha_2 x_2'}], \tag{2.8 contd.}$$

$$\breve{u}_2^{(I)}(x_2') = iu_0\eta_R^{-2}[2\alpha_1\,e^{-\omega s_R\alpha_1 x_2'} - \alpha_2^{-1}(2-\eta_R^2)\,e^{-\omega s_R\alpha_2 x_2'}],$$

$$\alpha_1 = (1-\epsilon^2\eta_R^2)^{1/2}, \alpha_2 = (1-\eta_R^2)^{1/2}, s_R = s_2/\eta_R,$$

$$\epsilon = s_1/s_2, s_1^2 = (1-2\nu)\rho_e[2\mu(1-\nu)]^{-1}, s_2^2 = \rho_e/\mu,$$

where η_R^2 is the root of the Rayleigh equation, $\eta^6 - 8\eta^4 + (24-16\epsilon^2)\eta^2 - 16(1-\epsilon^2) = 0$, that is real, positive and $0 < \eta^2 < 1$; ρ_e, μ and ν pertain to the elastic half-space, and u_0 is a specified scalar amplitude of the incident wave. Using eqns (2.4), (2.6) and (2.8) we obtain the theta averaged cylindrical components

$$\mathring{u}'_{r'}(r', z') = \mathring{u}_{r'}^{(S)}(r', z') + i\breve{u}_{1'}^{(I)}(z')J_1(\omega s_R r'),$$

$$\mathring{u}'_{z'}(r', z') = \mathring{u}_{z'}^{(S)}(r', z') + \breve{u}_{2'}^{(I)}(z')J_0(\omega s_R r'), \tag{2.9}$$

and

$$\lim_{r'\to\infty} (\mathring{u}_{r'}^{(S)}, \mathring{u}_{z'}^{(S)}) = \lim_{z'\to\infty} (\mathring{u}_{r'}^{(S)}, \mathring{u}_{z'}^{(S)}) = 0. \tag{2.10}$$

The problem formulation will exhibit several functions and physical parameters. The traction amplitudes on the crystal top surface ($z = b$), crystal bottom surface ($z = -b$), and test medium surface ($z' = 0$) are denoted as $\mathbf{T}^-(r, \theta, t), \mathbf{T}^+(r, \theta, t)$ and $\mathbf{T}'(r', \theta', t)$, respectively. The constants ρ_c, c_{IJ}, ϵ_{ij} and e_{iJ} pertain to the piezoelectric material of the crystal, as described in [7], the constants L, R and C are the inductance, resistance and capacitance of the elements in the transducer circuit, the constants m_s, c_s, k_s, m_n, c_n and k_n pertain to the viscoelastic foundation backing model, and u_2^{CG} and F_2^c are the components in the axial direction of the casing displacement and force resultant due to the tractions on the casing outer surface. M is the mass of the rigid casing. Before presenting the problem formulation, we define the following dimensionless quantities.

$$\tilde{r} = r/a = r'/a, \tilde{z} = z'/a, \tilde{u}_r^{(0)} = \mathring{u}_r^{(0)}/a, \tilde{u}_r^{(1)} = \mathring{u}_r^{(1)}/a, \tilde{u}_z^{(0)} = \mathring{u}_z^{(0)}/b,$$

$$\tilde{u}_z^{(1)} = \mathring{u}_z^{(1)}/b, \tilde{\phi}^{(1)} = (\pi/2b)(\epsilon_{22}/c_{44})^{1/2}\mathring{\phi}^{(1)}, \tilde{u}_r^{(S)} = \mathring{u}_{r'}^{(S)}/a,$$

$$\tilde{u}_z^{(S)} = \mathring{u}_{z'}^{(S)}/a, \tilde{u}_0 = u_0/a, \tilde{u}_2^{CG} = u_2^{CG}/b, \tilde{F} = (a/bc_{44})\mathring{T}_r^{\,-},$$

$$\tilde{H} = (a/bc_{44})\mathring{T}_r^{\,+}, \tilde{G} = (4/\pi^2 c_{44})\mathring{T}_z^{\,-}, \tilde{K} = (4/\pi^2 c_{44})\mathring{T}_z^{\,+},$$

$$\tilde{T}_r = \mathring{T}'_{r'}/\mu, \tilde{T}_z = \mathring{T}'_{z'}/\mu, \tilde{\omega} = (2b/\pi)(\rho_c/c_{44})^{1/2}\omega, \tilde{B} = (\epsilon_{22}/b^2c_{44})^{1/2}B,$$

$$\tilde{q} = a^{-2}(c_{44}\epsilon_{22})^{-1/2}q, \tilde{I} = (2b/a^2\pi c_{44})(\rho_c/\epsilon_{22})^{1/2}I, \tilde{E} = (\epsilon_{22}/b^2c_{44})^{1/2}E,$$

$$\tilde{L} = (\pi^2a^2c_{44}\epsilon_{22}/4b^3\rho_c)L, \tilde{R} = (\pi a^2\epsilon_{22}/2b^2)(c_{44}/\rho_c)^{1/2}R,$$

$$\tilde{C} = (b/a^2\epsilon_{22})C, \tilde{c}_{ij} = c_{ij}/c_{44}, \tilde{\epsilon}_{ij} = \epsilon_{ij}/\epsilon_{22}, \tag{2.11}$$

$$\tilde{e}_{ij} = (c_{44}\epsilon_{22})^{-(1/2)}e_{ij}, \tilde{\kappa} = (\pi a^2/2b^2)(\rho_c c_{44})^{-(1/2)}\kappa, \tilde{m}_s = (\tilde{\sigma}/b\rho_c)m_s,$$

$$\tilde{c}_s = (\pi a^2/2b^2)(\rho_c c_{44})^{-(1/2)}c_s, \tilde{k}_s = (a^2/bc_{44})k_s, \tilde{m}_n = (\rho_c b)^{-1}m_n,$$

$$\tilde{c}_n = (2/\pi)(\rho_c c_{44})^{-(1/2)}c_n, \tilde{k}_n = (4b/\pi^2c_{44})k_n, \tilde{F}_2^c = (\pi a^2c_{44})^{-1}F_2^c,$$

$$\tilde{M} = (\pi/4\rho_c a^2b)M, \tilde{\sigma} = (\pi a/2b)^2, \tilde{\gamma}_e^2 = \rho_e c_{44}/\rho_c\mu, \tilde{\delta}_e = \mu/c_{44},$$

$$\tilde{s}_1 = (\tilde{\sigma}c_{44}/\rho_c)^{1/2}s_1, \tilde{s}_2 = (\tilde{\sigma}c_{44}/\rho_c)^{1/2}s_2, \tilde{s}_R = (\tilde{\sigma}c_{44}/\rho_c)^{1/2}s_R.$$

Dropping the tildes, we state the nondimensional torsionless axisymmetric steady vibration problem as follows.

Crystal extensional

Equations of motion ($0 < r < 1$):

$$c_{11}\mathscr{B}_1(u_r^{(0)}) + \sigma\omega^2 u_r^{(0)} + \alpha c_{12} u_{z,r}^{(1)} = -(1/2)(F+H),$$
$$-8\alpha\sigma\pi^{-2}c_{12}r^{-1}(ru_r^{(0)})_{,r} + \mathscr{B}_0(u_z^{(1)}) + \sigma(\omega^2 - c_{22})u_z^{(1)} = \sigma(G-K) + 8\alpha\sigma\pi^{-2}e_{22}B, \quad (2.12)$$

in which $\mathscr{B}_0$ and $\mathscr{B}_1$ are the differential operators defined by

$$\mathscr{B}_k = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{k}{r^2}, k = 0, 1. \quad (2.13)$$

Transducer charge, current and circuit equation:

$$q = 2\pi e_{21}u_r^{(0)}(1) + \pi^2 e_{22}\int_0^1 u_z^{(1)}(r)r\,\mathrm{d}r - \pi B,$$
$$I = -i\omega q, (-\omega^2 L - i\omega R + C^{-1})q = E + 2B. \quad (2.14)$$

Boundary conditions:

$$u_r^{(0)}(0) = u_r^{(0)}(1) = u_{z,r}^{(1)}(0) = u_{z,r}^{(1)}(1) = 0. \quad (2.15)$$

Crystal flexural

Equations of motion and charge equation ($0 < r < 1$):

$$c_{11}\mathscr{B}_1(u_r^{(1)}) + \sigma(\omega^2 - 1)u_r^{(1)} - 2\alpha u_{z,r}^{(0)} - (e_{16} + e_{21})\phi_{,r}^{(1)} = F - H,$$
$$4\alpha\sigma\pi^{-2}r^{-1}(ru_r^{(1)})_{,r} + \mathscr{B}_0(u_z^{(0)}) + \sigma\omega^2 u_z^{(0)} + 4\alpha\pi^{-2}e_{16}\mathscr{B}_0(\phi^{(1)}) = -(\sigma/2)(G+K), \quad (2.16)$$
$$\sigma(e_{16} + e_{21})r^{-1}(ru_r^{(1)})_{,r} + 2\alpha e_{16}\mathscr{B}_0(u_z^{(0)}) - \epsilon_{11}\mathscr{B}_0(\phi^{(1)}) + \sigma\phi^{(1)} = 0.$$

Boundary conditions:

$$u_r^{(1)}(0) = u_r^{(1)}(1) = u_{z,r}^{(0)}(0) = \phi_{,r}^{(1)}(0) = 0,$$
$$2\alpha e_{16}u_{z,r}^{(0)}(1) - \epsilon_{11}\phi_{,r}^{(1)}(1) = 0, \quad (2.17)$$
$$(\pi^2/4)u_{z,r}^{(0)}(1) + \alpha e_{21}\phi_{,r}^{(1)}(1) = 0.$$

In [11] it is concluded that, when modeling the typical commercial transducer unit, the selection of crystal mechanical edge conditions can be made on the basis of convenience of analysis rather than on the basis of apparent transducer design, since changes in them had little effect on the backed transducer's output. The edge condition which results in the simplest analysis was seen in [12] to be that of smooth edge contact with the casing, i.e. the crystal lateral boundary is constrained by the smooth, rigid casing surface $r = a$, $0 \leqslant \theta < 2\pi$, $-b < z < b$. The boundary conditions, eqns $(2.15)_{2,4}$ and $(2.17)_{2,6}$ result from that choice (in [12] the crystal mechanical edge conditions of traction free edge, crystal edge simply supported by casing and crystal edge clamped to casing are also considered). Equation $(2.17)_5$ is a consequence of the assumption that the crystal lateral surface is electrically insulated.

Elastic half-space scattered wave

Equations of motion ($0 < r, z < \infty$):

$$
\begin{aligned}
&2(1-\nu)(1-2\nu)^{-1}\mathscr{B}_1(u_r^{(S)}) + u_{r,zz}^{(S)} + \sigma\gamma_e{}^2\omega^2 u_r^{(S)} + (1-2\nu)^{-1}u_{z,rz}^{(S)} = 0,\\
&(1-2\nu)^{-1}r^{-1}(ru_r^{(S)})_{,rz} + \mathscr{B}_0(u_z^{(S)}) + 2(1-\nu)(1-2\nu)^{-1}u_{z,zz}^{(S)} + \sigma\gamma_e{}^2\omega^2 u_z^{(S)} = 0.
\end{aligned} \tag{2.18}
$$

Boundary conditions:

$$
u_r^{(S)}(0,z) = u_{z,r}^{(S)}(0,z) = 0, \tag{2.19}
$$

$$
\lim_{r\to\infty}(u_r^{(S)}, u_z^{(S)}) = \lim_{z\to\infty}(u_r^{(S)}, u_z^{(S)}) = 0,
$$

and at $z = 0$:

$$
u_{r,z}^{(S)} + u_{z,r}^{(S)} = \begin{cases} -T_r & r < 1 \\ 0 & r > 1 \end{cases},
$$

$$
\nu r^{-1}(ru_r^{(S)})_{,r} + (1-\nu)u_{z,z}^{(S)} = \begin{cases} -2^{-1}(1-2\nu)T_z & r < 1 \\ 0 & r > 1 \end{cases}. \tag{2.20}
$$

Center conditions, eqns $(2.15)_{1,3}$, $(2.17)_{1,3}$ and $(2.19)_{1,2}$ follow from eqns (2.2) and (2.4) and the requirements that the crystal and the half-space remain simply connected and do not form a cusp at the origin. The center condition eqn $(2.17)_4$ follows from eqn (2.4) and the condition that the divergence of the electric displacement vector is zero at the origin.

Casing rigid body motion

$$
u_2^{CG} = (M\omega^2)^{-1}\left[(\pi^2/2)\int_0^1 K(r)r\,\mathrm{d}r - F_2^c\right], \tag{2.21}
$$

with F_2^c the specified amplitude of an axial force applied to the casing. Only the u_2^{CG} component of the casing rigid body motion enters in the $B(t)$ problem.

Crystal/elastic half-space interface conditions $(0 < r < 1)$

$$
F(r) + i\omega\kappa[-u_r^{(0)}(r) + u_r^{(1)}(r) + u_r^{(S)}(r,0)] = \omega\kappa u_0 J_1(\omega s_R r), \tag{2.22}
$$

$$
u_z^{(0)}(r) - u_z^{(1)}(r) + (2/\pi)\sigma^{\frac{1}{2}}u_z^{(S)}(r,0) = -(2/\pi)\sigma^{1/2}i[2\eta_R^{-2}(\alpha_1 - \alpha_2^{-1}) + \alpha_2^{-1}]J_0(\omega s_R r)u_0,
$$

$$
F(r) = -(2/\pi)\sigma^{1/2}\delta_e T_r(r),\; G(r) = (4/\pi^2)\delta_e T_z(r).
$$

Equation (2.22) is the transducer/test medium coupling model of [15] as described earlier in this section.

Crystal/backing interface conditions $(0 < r < 1)$

$$
\begin{aligned}
H(r) &= (\omega^2 m_s + i\omega c_s - k_s)[u_r^{(0)}(r) + u_r^{(1)}(r)],\\
K(r) &= (\omega^2 m_n + i\omega c_n - k_n)[u_z^{(0)}(r) + u_z^{(1)}(r) - u_2^{CG}].
\end{aligned} \tag{2.23}
$$

Equation (2.23) reflects the viscoelastic foundation backing model considered in [12].

In obtaining the analytical solution to the nondimensional transducer voltage problem, eqns (2.12)–(2.23), the following scheme is utilized, so as to treat separately the crystal extensional, crystal flexural and elastic half-space problems for as long as possible. The crystal extensional deformations, transducer circuit current and transducer voltage are solved for in terms of the unknown top and bottom surface tractions F, G, H and K, and the specified voltage of E. The crystal flexural deformations are found in terms of the unknown tractions F, G, H and K. The solution for the scattered wave field is given in terms of the interface tractions T_r and T_z. (The casing motion is already expressed by eqn (2.21) in terms of K and the specified resultant force F_2^c.) Finally, we apply the interface conditions at the crystal top and bottom surfaces, eqns (2.22) and (2.23), to obtain coupled integral equations for the interface tractions F, G, H, K, T_r and T_z, in terms of the specified quantities E, F_2^c and u_0.

We use the theory of Bessel functions to obtain the solution (see [8]). Let λ_n, $n = 0, 1, 2, \ldots$, be the nonnegative roots of

$$J_1(\lambda_n) = 0, \tag{2.24}$$

and let $\mathscr{F}(r)$and $\mathscr{G}(r)$ be any functions suitably defined on $0 < r < 1$, with

$$\mathscr{F}(0) = \mathscr{F}(1) = \mathscr{G}_{,r}(0) = \mathscr{G}_{,r}(1) = 0. \tag{2.25}$$

Then

$$\mathscr{F}(r) = 2\sum_{\lambda_n > 0} \bar{\mathscr{F}}(\lambda_n)\frac{J_1(\lambda_n r)}{J_0^2(\lambda_n)}, \quad \mathscr{G}(r) = 2\sum_{\lambda_n \ge 0} \hat{\mathscr{G}}(\lambda_n)\frac{J_0(\lambda_n r)}{J_0^2(\lambda_n)}, \tag{2.26}$$

with

$$\bar{\mathscr{F}}(\lambda_n) = \int_0^1 \mathscr{F}(r) r J_1(\lambda_n r)\,\mathrm{d}r \equiv \mathscr{H}_1[\mathscr{F}; r \to \lambda_n],$$

$$\hat{\mathscr{G}}(\lambda_n) = \int_0^1 \mathscr{G}(r) r J_0(\lambda_n r)\,\mathrm{d}r \equiv \mathscr{H}_0[\mathscr{G}; r \to \lambda_n]. \tag{2.27}$$

Also

$$\mathscr{H}_1[\mathscr{B}_1(\mathscr{F}); r \to \lambda_n] = -\lambda_n^2 \bar{\mathscr{F}}(\lambda_n), \; \mathscr{H}_1[\mathscr{G}_{,r}; r \to \lambda_n] = -\lambda_n \hat{\mathscr{G}}(\lambda_n),$$

$$\mathscr{H}_1[r; r \to \lambda_n] = \begin{cases} -\lambda_n^{-1} J_0(\lambda_n) & \lambda_n > 0 \\ 0 & \lambda_n = 0 \end{cases},$$

$$\mathscr{H}_0[\mathscr{B}_0(\mathscr{G}); r \to \lambda_n] = -\lambda_n^2 \hat{\mathscr{G}}(\lambda_n), \; \mathscr{H}_0[r^{-1}(r\mathscr{F})_{,r}; r \to \lambda_n] = \lambda_n \bar{\mathscr{F}}(\lambda_n), \tag{2.28}$$

$$\mathscr{H}_0[r^2; r \to \lambda_n] = \begin{cases} 2\lambda_n^{-2} J_0(\lambda_n) & \lambda_n > 0 \\ 1/4 & \lambda_n = 0 \end{cases},$$

$$\mathscr{H}_0[1; r \to \lambda_n] = \begin{cases} 0 & \lambda_n > 0 \\ 1/2 & \lambda_n = 0 \end{cases}.$$

In view of eqns (2.15) and (2.25)–(2.26) we may write for the extensional problem

$$u_r^{(0)}(r) = 2\sum_{\lambda_n > 0} \bar{u}_r^{(0)}(\lambda_n)\frac{J_1(\lambda_n r)}{J_0^2(\lambda_n)}, \; u_z^{(1)}(r) = 2\sum_{\lambda_n \geq 0} \hat{u}_z^{(1)}(\lambda_n)\frac{J_0(\lambda_n r)}{J_0^2(\lambda_n)}. \tag{2.29}$$

Next we apply Hankel transforms as indicated by $\mathscr{H}_1$ [eqn (2.12)$_1$; $r \to \lambda_n > 0$] and $\mathscr{H}_0$ [eqn (2.12)$_2$; $r \to \lambda_n > 0$] and, solving the resulting equations, we obtain for $\lambda_n > 0$

$$\begin{aligned} \bar{u}_r^{(0)}(\lambda_n) &= \{\theta_1^E(\lambda_n)[\bar{F}(\lambda_n) + \bar{H}(\lambda_n)] + \theta_2^E(\lambda_n)[\hat{G}(\lambda_n) - \hat{K}(\lambda_n)]\}/\psi^E(\lambda_n), \\ \hat{u}_z^{(1)}(\lambda_n) &= \{\phi_1^E(\lambda_n)[\bar{F}(\lambda_n) + \bar{H}(\lambda_n)] + \phi_2^E(\lambda_n)[\hat{G}(\lambda_n) - \hat{K}(\lambda_n)]\}/\psi^E(\lambda_n), \end{aligned} \tag{2.30}$$

where

$$\begin{aligned}
&\psi^E(\lambda_n) = \omega^4\sigma^2 c_{11}^{-1} - \omega^2\sigma[\lambda_n^2(1 + c_{11}^{-1}) + \sigma c_{22}c_{11}^{-1}] + \lambda_n^2 l(\lambda_n),\\
&\theta_1^E(\lambda_n) = (-\omega^2\sigma + \lambda_n^2 + \sigma c_{22})/2c_{11},\ \theta_2^E(\lambda_n) = \lambda_n \alpha c_{12}\sigma/c_{11},\\
&\phi_1^E(\lambda_n) = -4\alpha\sigma c_{12}\lambda_n/\pi^2 c_{11},\ \phi_2^E(\lambda_n) = \sigma(\omega^2\sigma/c_{11} - \lambda_n^2),\\
&l(\lambda_n) = \lambda_n^2 + \sigma c_{22} - 8\alpha^2\sigma c_{12}^2/\pi^2 c_{11}.
\end{aligned} \tag{2.31}$$

Also, the Hankel transform $\mathscr{H}_0$ [eqn $(2.12)_2$; $r \to \lambda_0 = 0$] and eqn (2.14) are solved to yield

$$\hat{u}_r^{(1)}(0) = \{\gamma_2^E[\hat{G}(0) - \hat{K}(0)] + \gamma_5^E E\}/\Delta,\ B = \{\beta_2[\hat{G}(0) - \hat{K}(0)] + \beta_5 E\}/\Delta, \tag{2.32}$$

with

$$\begin{aligned}
\Delta = d(\omega^2 - c_{22} - 4\alpha e_{22}^2\pi^{-1}) + 2(\omega^2 - c_{22})\pi^{-1},\ \gamma_2^E = d + 2\pi^{-1},\\
\gamma_5^E = -4\alpha\pi^{-3}e_{22},\ \beta_2 = d\pi e_{22},
\end{aligned} \tag{2.33}$$

$$\beta_5 = (-\omega^2 + c_{22})\pi^{-1},\ d = -\omega^2 L - i\omega R + C^{-1}.$$

From eqns (2.29)–(2.33) we obtain for the extensional problem

$$u_r^{(0)}(r) = \sum_{\lambda_n>0}\left[\frac{2J_1(\lambda_n r)\theta_1^E(\lambda_n)}{J_0^2(\lambda_n)\psi^E(\lambda_n)}\right][\hat{F}(\lambda_n) + \hat{H}(\lambda_n)] + \sum_{\lambda_n>0}\left[\frac{2J_1(\lambda_n r)\theta_2^E(\lambda_n)}{J_0^2(\lambda_n)\psi^E(\lambda_n)}\right][\hat{G}(\lambda_n) - \hat{K}(\lambda_n)],$$

$$u_z^{(1)}(r) = \sum_{\lambda_n>0}\left[\frac{2J_0(\lambda_n r)\phi_1^E(\lambda_n)}{J_0^2(\lambda_n)\psi^E(\lambda_n)}\right][\hat{F}(\lambda_n) + \hat{H}(\lambda_n)] + \frac{2\gamma_2^E}{\Delta}[\hat{G}(0) - \hat{K}(0)]$$

$$+ \sum_{\lambda_n>0}\left[\frac{2J_0(\lambda_n r)\phi_2^E(\lambda_n)}{J_0^2(\lambda_n)\psi^E(\lambda_n)}\right][\hat{G}(\lambda_n) - \hat{K}(\lambda_n)] + \frac{2\gamma_5^E}{\Delta}E, \tag{2.34}$$

$$B = \beta_2\Delta^{-1}[\hat{G}(0) - \hat{K}(0)] + \beta_5\Delta^{-1}E, \qquad I = -(i\omega d^{-1})(2B + E).$$

Recall that the primary goal of the steady vibration problem under consideration is to express the transducer voltage amplitude $2B$ and the transducer circuit current amplitude I as functions of the specified incident surface wave amplitude u_0, impressed voltage amplitude E, resultant amplitude F_z^c of the traction on the casing outer surface, and frequency of vibration ω. In eqn (2.34) we see that B and I may be expressed completely in terms of E and $[\hat{G}(0) - \hat{K}(0)]$, which is the non-dimensional form of the "compressional" resultant amplitude, i.e.

$$-\int_0^a\int_0^{2\pi}[T_{zz}(r,\theta,b) + T_{zz}(r,\theta,-b)]r\,\mathrm{d}r\mathrm{d}\theta = (\pi^3 a^2 c_{44}/2)[\hat{G}(0) - \hat{K}(0)]. \tag{2.35}$$

This has been accomplished by considering only the crystal extensional problem. However, the determination of $[\hat{G}(0) - \hat{K}(0)]$ will require consideration of the complete torsionless axisymmetric problem, i.e. the crystal flexural, elastic half-space and casing motion problems and the interface conditions, in addition to the crystal extensional problem.

In a manner similar to that above, we obtain the crystal flexural displacement components from eqns (2.16) and (2.17).

$$u_r^{(1)}(r) = \sum_{\lambda_n>0}\left[\frac{2J_1(\lambda_n r)\theta_1^F(\lambda_n)}{J_0^2(\lambda_n)\psi^F(\lambda_n)}\right][\hat{F}(\lambda_n) - \hat{H}(\lambda_n)] + \sum_{\lambda_n>0}\left[\frac{2J_1(\lambda_n r)\theta_2^F(\lambda_n)}{J_0^2(\lambda_n)\psi^F(\lambda_n)}\right][\hat{G}(\lambda_n) + \hat{K}(\lambda_n)],$$

$$u_z^{(0)}(r) = \sum_{\lambda_n>0}\left[\frac{2J_0(\lambda_n r)\phi_1^F(\lambda_n)}{J_0^2(\lambda_n)\psi^F(\lambda_n)}\right][\hat{F}(\lambda_n) - \hat{H}(\lambda_n)] + 2\gamma_2^F[\hat{G}(0) + \hat{K}(0)] \tag{2.36}$$

$$+ \sum_{\lambda_n>0}\left[\frac{2J_0(\lambda_n r)\phi_2^F(\lambda_n)}{J_0^2(\lambda_n)\psi^F(\lambda_n)}\right][\hat{G}(\lambda_n) + \hat{K}(\lambda_n)]$$

with

$$\begin{aligned}
\gamma_2{}^F &= -(2\omega^2)^{-1}, \\
\psi^F(\lambda_n) &= \lambda_n{}^6 c_{11}(\epsilon_{11} + 8\alpha^2\pi^{-2}e_{16}^2) + \lambda_n{}^4\sigma\{-\omega^2[\epsilon_{11}(1+c_{11}) + 8\alpha^2\pi^{-2}e_{16}^2] \\
&\quad + c_{11} + \epsilon_{11} + (e_{16}+e_{21})^2 - 8\alpha^2\pi^{-2}(\epsilon_{11} + e_{16}^2 + 2e_{16}e_{21})\} \\
&\quad + \lambda_n{}^2\sigma^2\{\epsilon_{11}\omega^4 - \omega^2[\epsilon_{11} + c_{11} + 1 + (e_{16}+e_{21})^2] + 1 - 8\alpha^2\pi^{-2}\} + \sigma^3(\omega^4 - \omega^2), \\
\theta_1{}^F(\lambda_n) &= \omega^2\sigma(\epsilon_{11}\lambda_n{}^2 + \sigma) - \lambda_n{}^4(\epsilon_{11} + 8\alpha^2\pi^{-2}e_{16}^2) - \lambda_n{}^2\sigma, \qquad (2.37) \\
\theta_2{}^F(\lambda_n) &= \sigma\alpha\lambda_n{}^3[\epsilon_{11} + e_{16}(e_{16}+e_{21})] + \sigma^2\alpha\lambda_n, \\
\phi_1{}^F(\lambda_n) &= -4\alpha\sigma\pi^{-2}\{\lambda_n{}^3[\epsilon_{11} + e_{16}(e_{16}+e_{21})] + \lambda_n\sigma\}, \\
\phi_2{}^F(\lambda_n) &= (\sigma/2)\{-\omega^2(\sigma\lambda_n{}^2\epsilon_{11} + \sigma^2) + \lambda_n{}^4\epsilon_{11}c_{11} + \lambda_n{}^2\sigma[c_{11} + \epsilon_{11} + (e_{16}+e_{21})^2] + \sigma^2\}.
\end{aligned}$$

The solution to the elastic scattered wave problem, eqns (2.18)–(2.20), can be found in many sources, e.g. [12], and in particular we may express

$$u_r^{(S)}(r,0) = \int_0^\infty [\mathscr{R}_R^{(S)}(\xi,r)\hat{T}_r(\xi) + \mathscr{R}_Z^{(S)}(\xi,r)\hat{T}_z(\xi)]\,\mathrm{d}\xi,$$

$$(2.38)$$

$$u_r^{(S)}(r,0) = \int_0^\infty [\mathscr{Z}_R^{(S)}(\xi,r)\hat{T}_r(\xi) + \mathscr{Z}_Z^{(S)}(\xi,r)\hat{T}_z(\xi)]\,\mathrm{d}\xi,$$

with

$$\begin{aligned}
&\mathscr{R}_R^{(S)}(\xi,r) = -\omega^2 s_2{}^2 v_2\xi J_1(\xi r)/R(\xi), \quad \mathscr{Z}_Z^{(S)}(\xi,r) = -\omega^2 s_2{}^2 v_1\xi J_0(\xi r)/R(\xi), \\
&\mathscr{R}_Z^{(S)}(\xi,r) = \xi^2(2\xi^2 - 2v_1v_2 - \omega^2 s_2{}^2)J_1(\xi r)/R(\xi), \\
&\mathscr{Z}_R^{(S)}(\xi,r) = \xi^2(2\xi^2 - 2v_1v_2 - \omega^2 s_2{}^2)J_0(\xi r)/R(\xi), \qquad (2.39) \\
&\quad R(\xi) = (2\xi^2 - \omega^2 s_2{}^2)^2 - 4v_1v_2\xi^2;\ v_\alpha{}^2 = \xi^2 - \omega^2 s_\alpha{}^2,\ \mathrm{Re}\,\{v_\alpha\} \ge 0.
\end{aligned}$$

At present we have the crystal extensional and flexural, casing, and elastic half-space solutions in terms of the prescribed amplitudes $F_2{}^c$ and E, and the unknown traction components $F(r)$, $H(r)$, $G(r)$, $K(r)$, $T_r(r)$ and $T_z(r)$. We now use the interface conditions, eqns (2.22) and (2.23), to obtain integral equations for the unknown traction components in terms of the known forcings u_0, $F_2{}^c$ and E.

Applications of the Hankel transforms $\mathscr{H}_1$ [eqn $(2.22)_3$; $r\to\xi$] and $\mathscr{H}_0$ [eqn $(2.22)_4$; $r\to\xi$] give

$$\hat{T}_r(\xi) = -(\pi/2\sigma^{1/2}\delta_e)\hat{F}(\xi),\ \hat{T}_z(\xi) = (\pi^2/4\delta_e)\hat{G}(\xi). \qquad (2.40)$$

Using eqns (2.21), (2.34), (2.36) and (2.38)–(2.40), we obtain from eqns $(2.22)_{1,2}$ and $(2.23)_{1,2}$ the system of integral equations

$$(1-\delta_{i2})\Psi_i(r) + \int_0^1 \sum_{j=1}^4 K_{ij}(r,\rho)\Psi_j(\rho)\,\mathrm{d}\rho = \Lambda_i(r), \quad 0<r<1,\ i=1,2,3,4, \qquad (2.41)$$

with

$$\Psi_1(r) = F(r),\ \Psi_2(r) = G(r),\ \Psi_3(r) = H(r),\ \Psi_4(r) = K(r), \qquad (2.42)$$

and

$$K_{11}(r,\rho) = -i\omega\kappa \sum_{\lambda_n>0} J_1(\lambda_n r) J_1(\lambda_n \rho)\rho R_1^-(\lambda_n)$$
$$+ (i\omega^3 s_2^2 v_2 \kappa\pi/2\sigma^{1/2}\delta_e) \int_0^\infty J_1(\xi r) J_1(\xi\rho)\rho[\xi/R(\xi)]\,d\xi,$$

$$K_{12}(r,\rho) = -i\omega\kappa \sum_{\lambda_n>0} J_1(\lambda_n r) J_0(\lambda_n \rho)\rho R_2^-(\lambda_n)$$
$$+ (i\omega\kappa\pi^2/4\delta_e) \int_0^\infty J_1(\xi r) J_0(\xi\rho)\rho[\xi^2(2\xi^2 - 2v_1 v_2 - \omega^2 s_2^2)/R(\xi)]\,d\xi,$$

$$K_{13}(r,\rho) = -i\omega\kappa \sum_{\lambda_n>0} J_1(\lambda_n r) J_1(\lambda_n \rho)\rho R_1^+(\lambda_n),$$

$$K_{14}(r,\rho) = -i\omega\kappa \sum_{\lambda_n>0} J_1(\lambda_n r) J_0(\lambda_n \rho)\rho R_2^+(\lambda_n),$$

$$\Lambda_1(r) = \omega\kappa J_1(\omega s_R r) u_0,$$

$$K_{21}(r,\rho) = -\sum_{\lambda_n>0} J_0(\lambda_n r) J_1(\lambda_n \rho)\rho Z_1^-(\lambda_n)$$
$$-\delta_e^{-1} \int_0^\infty J_0(\xi r) J_1(\xi\rho)\rho[\xi^2(2\xi^2 - 2v_1 v_2 - \omega^2 s_2^2)/R(\xi)]\,d\xi,$$

$$K_{22}(r,\rho) = 2\rho(-\gamma_2^E/\Delta + \gamma_2^F) - \sum_{\lambda_n>0} J_0(\lambda_n r) J_0(\lambda_n \rho)\rho Z_2^-(\lambda_n) - (\pi\sigma^{1/2}\omega^2 s_2^2 v_1/2\delta_e)$$
$$\times \int_0^\infty J_0(\xi r) J_0(\xi\rho)\rho[\xi/R(\xi)]\,d\xi, \tag{2.43}$$

$$K_{23}(r,\rho) = -\sum_{\lambda_n>0} J_0(\lambda_n r) J_1(\lambda_n \rho)\rho Z_1^+(\lambda_n),$$

$$K_{24}(r,\rho) = 2\rho(\gamma_2^E/\Delta + \gamma_2^F) + \sum_{\lambda_n>0} J_0(\lambda_n r) J_0(\lambda_n \rho)\rho Z_2^+(\lambda_n),$$

$$\Lambda_2(r) = -(2\sigma^{1/2} i/\pi)[2\eta_R^{-2}(\alpha_1 - \alpha_2^{-1}) + \alpha_2^{-1}] J_0(\omega s_R r) u_0 + 2\gamma_5^E \Delta^{-1} E,$$

$$K_{31}(r,\rho) = d_s \sum_{\lambda_n>0} J_1(\lambda_n r) J_1(\lambda_n \rho)\rho R_1^+(\lambda_n),$$

$$K_{32}(r,\rho) = d_s \sum_{\lambda_n>0} J_1(\lambda_n r) J_0(\lambda_n \rho)\rho R_2^+(\lambda_n),$$

$$K_{33}(r,\rho) = d_s \sum_{\lambda_n>0} J_1(\lambda_n r) J_1(\lambda_n \rho)\rho R_1^-(\lambda_n),$$

$$K_{34}(r,\rho) = -d_s \sum_{\lambda_n>0} J_1(\lambda_n r) J_0(\lambda_n \rho)\rho R_2^-(\lambda_n),$$

$$\Lambda_3(r) = 0,$$

$$K_{41}(r,\rho) = d_n \sum_{\lambda_n>0} J_0(\lambda_n r) J_1(\lambda_n \rho)\rho Z_1^+(\lambda_n),$$

$$K_{42}(r,\rho)=2d_n\rho(\gamma_2^E/\Delta+\gamma_2^F)+d_n\sum_{\lambda_n>0}J_0(\lambda_n r)J_0(\lambda_n\rho)Z_1^+(\lambda_n),$$

$$K_{43}(r,\rho)=d_n\sum_{\lambda_n>0}J_0(\lambda_n r)J_1(\lambda_n\rho)\rho Z_1^-(\lambda_n), \qquad \text{(2.43) contd.}$$

$$K_{44}(r,\rho)=2d_n\rho(-\gamma_2^E/\Delta+\gamma_2^F-\pi^2/4M\omega^2)-d_n\sum_{\lambda_n>0}J_0(\lambda_n r)J_0(\lambda_n\rho)\rho Z_2^-(\lambda_n),$$

$$A_4(r)=-d_n[2\lambda_5^E/\Delta]E-d_n(M\omega^2)^{-1}F_2^c.$$

In eqn (2.41) we have used the notation

$$\delta_{22}=1;\qquad \delta_{i2}=0,\qquad i=1,3,4, \qquad (2.44)$$

and in eqn (2.43)

$$R_1^\pm(\lambda_n)=\frac{2}{J_0^2(\lambda_n)}\left[\frac{\theta_1^E(\lambda_n)}{\psi^E(\lambda_n)}\pm\frac{\theta_1^F(\lambda_n)}{\psi^F(\lambda_n)}\right],\ R_2^\pm(\lambda_n)=\frac{2}{J_0^2(\lambda_n)}\left[\frac{\theta_2^E(\lambda_n)}{\psi^E(\lambda_n)}\pm\frac{\theta_2^F(\lambda_n)}{\psi^F(\lambda_n)}\right],$$

$$Z_1^\pm(\lambda_n)=\frac{2}{J_0^2(\lambda_n)}\left[\frac{\phi_1^E(\lambda_n)}{\psi^E(\lambda_n)}\pm\frac{\phi_1^F(\lambda_n)}{\psi^F(\lambda_n)}\right],\ Z_2^\pm(\lambda_n)=\frac{2}{J_0^2(\lambda_n)}\left[\frac{\phi_2^E(\lambda_n)}{\psi^E(\lambda_n)}\pm\frac{\phi_2^F(\lambda_n)}{\psi^F(\lambda_n)}\right], \qquad (2.45)$$

$$d_s=-\omega^2m_s-i\omega c_s+k_s,\ d_n=-\omega^2m_n-i\omega c_n+k_n.$$

Equations $(2.41)_{1-4}$ are four coupled integral equations for determining $F(r)$, $G(r)$, $H(r)$ and $K(r)$ on $0<r<1$. From the solution of eqn (2.41), $T_r(r)$ and $T_z(r)$ are obtained via eqns $(2.22)_{3,4}$.

If eqns $(2.41)_{1-4}$ are solved for $F(r)$, $G(r)$, $H(r)$ and $K(r)$ in terms of u_0, E and F_2^c, then by use of eqn (2.34), we may express the transducer voltage and current amplitudes $2B$ and I in terms of these quantities. Likewise, the axisymmetric torsionless components of the scattered wave in the elastic half-space, the casing displacements and the crystal displacements can be expressed in terms of u_0, E and F_2^c by the use of eqns (2.21), (2.34), (2.36), (2.38) and (2.40). Hence the torsionless axisymmetric steady vibration solution is complete once eqns $(2.41)_{1-4}$ are solved.

The system of integral equations, eqn (2.41), can be solved in a manner similar to that employed in [15]. These equations are also undoubtedly singular, and their algebraic complexity makes their numerical solutions quite tedious and expensive. Rather than pursuing their solution we have chosen to consider plausible simplifying assumptions which allow us to get some of the desired results without solving the integral equations.

3. APPROXIMATIONS ON THE TIME HARMONIC PROBLEM

In this section we present some approximations to the problem formulation of Section 2 in order to simplify the integral equations, eqn (2.41). We give two approximations of the crystal/test medium interaction. They alter only the crystal/elastic half-space interface conditions, eqn (2.22), and not the crystal/backing interface conditions, eqn (2.23). Hence, they alter the system of integral equations, eqn (2.41), only for $i=1, 2$, and leave the equations unchanged for $i=3, 4$.

In the complete problem, the presence of the transducer on the test medium surface causes the scattering of the incident wave in the test medium and nonzero tractions at the interface between the transducer and test medium. The first approximation neglects the scattered wave part of the test medium displacement, while the second involves the postulating of the interface tractions. Both circumvent the difficult modeling of the transducer/test medium coupling.

Prescribed displacements on the crystal bottom surface

This approximation simplifies the integral equations, eqns $(2.41)_{1,2}$, by removing the integral terms in the kernels. This allows an explicit representation of the quantities B, I, $u_r^{(0)}(r)$, $u_r^{(1)}(r)$, $u_z^{(0)}(r)$, $u_z^{(1)}(r)$ and u_2^{CG} as functions of u_0, F_2^c, E, ω and the various material parameters.

First we assume the scattered wave part of the elastic half-space displacement is zero. With this assumption, eqns $(2.22)_{1,2}$ become

$$u_r^{(0)}(r) - u_r^{(1)}(r) = iu_0 J_1(\omega s_R r),$$

$$u_z^{(0)}(r) - u_z^{(1)}(r) = -(2/\pi)\sigma^{1/2} iu_0[2\eta_R^{-2}(\alpha_1 - \alpha_2^{-1}) + \alpha_2^{-1}]J_0(\omega s_R r), \tag{3.1}$$

and eqns $(2.22)_{3,4}$ become

$$F(r) = 0, G(r) = 0. \tag{3.2}$$

The conditions, eqns (3.1) and (3.2), together overspecify the first order plate theory crystal problem. We must therefore use only one of the following four pairs of crystal bottom surface conditions: eqns $(3.1)_{1,2}$, eqns $(3.2)_{1,2}$, eqns $(3.1)_1$ and $(3.2)_2$, or eqns $(3.1)_2$ and $(3.2)_1$. The second choice is clearly inadequate, since with it we would lose the incident surface wave nature of the problem. The last two, mixed, choices might also be worthy of study, but we pursue here only the first choice, i.e. we specify the crystal bottom surface displacements.

We replace eqns $(2.22)_{1,2}$ with eqn (3.1) and abandon eqns $(2.22)_{3,4}$. The integral equation $(2.41)_{1,2}$ are then replaced by

$$\int_0^1 \sum_{j=1}^4 K_{ij}^d(r,\rho)\Psi_j(\rho)\,d\rho = \Lambda_i^d(r), 0 < r < 1, i = 1, 2, \tag{3.3}$$

in which the kernel functions now have no integral terms, and are given by

$$K_{11}^d(r,\rho) = \sum_{\lambda_n > 0} J_1(\lambda_n r)J_1(\lambda_n \rho)\rho R_1^-(\lambda_n),$$

$$K_{12}^d(r,\rho) = \sum_{\lambda_n > 0} J_1(\lambda_n r)J_0(\lambda_n \rho)\rho R_2^-(\lambda_n),$$

$$K_{13}^d(r,\rho) = \sum_{\lambda_n > 0} J_1(\lambda_n r)J_1(\lambda_n \rho)\rho R_1^+(\lambda_n),$$

$$K_{14}^d(r,\rho) = \sum_{\lambda_n > 0} J_1(\lambda_n r)J_0(\lambda_n \rho)\rho R_2^+(\lambda_n),$$

$$\Lambda_1^d(r) = iu_0 J_1(\omega s_R r), K_{21}^d(r,\rho) = -\sum_{\lambda_n > 0} J_0(\lambda_n r)J_1(\lambda_n \rho)\rho Z_1^-(\lambda_n), \tag{3.4}$$

$$K_{22}^d(r,\rho) = -2\rho[(\gamma_2^E/\Delta) - \gamma_2^F] - \sum_{\lambda_n > 0} J_0(\lambda_n r)J_0(\lambda_n \rho)\rho Z_2^-(\lambda_n),$$

$$K_{23}^d(r,\rho) = -\sum_{\lambda_n > 0} J_0(\lambda_n r)J_1(\lambda_n \rho)\rho Z_1^+(\lambda_n),$$

$$K_{24}^d(r,\rho) = 2\rho[(\gamma_2^E/\Delta) + \gamma_2^F] + \sum_{\lambda_n > 0} J_0(\lambda_n r)J_0(\lambda_n \rho)\rho Z_2^+(\lambda_n),$$

$$\Lambda_2^d(r) = -(2/\pi)\sigma^{1/2} iu_0[2\eta_R^{-2}(\alpha_1 - \alpha_2^{-1}) + \alpha_2^{-1}]J_0(\omega s_R r) + (2\gamma_5^E/\Delta)E.$$

The integral equations, eqns $(2.41)_{3,4}$, remain unchanged. The problem for the determination of the interface traction components $F(r)$, $G(r)$, $H(r)$ and $K(r)$ has now been reduced to eqns $(2.41)_{3,4}$ and $(3.3)_{1,2}$, together with eqns (2.42), $(2.43)_{11\text{-}20}$ and (3.4). Only summation terms occur in the kernels, and with the use of eqn (2.27) the problem can be written as follows, for $0 < r < 1$:

$$\sum_{\lambda_n>0} J_1(\lambda_n r)[R_1^-(\lambda_n)\hat{F}(\lambda_n) + R_2^-(\lambda_n)\hat{G}(\lambda_n) + R_1^+(\lambda_n)\hat{H}(\lambda_n) - R_2^+(\lambda_n)\hat{K}(\lambda_n)] = iu_0 J_1(\omega s_R r),$$

$$2[(\gamma_2^E/\Delta) - \gamma_2^F]\hat{G}(0) - 2[(\gamma_2^E/\Delta) + \gamma_2^F]\hat{K}(0) + \sum_{\lambda_n>0} J_0(\lambda_n r)[Z_1^-(\lambda_n)\hat{F}(\lambda_n) + Z_2^-(\lambda_n)\hat{G}(\lambda_n)$$
$$+ Z_1^+(\lambda_n)\hat{H}(\lambda_n) - Z_2^+(\lambda_n)\hat{K}(\lambda_n)] = (2/\pi)\sigma^{1/2} iu_0 J_0(\omega s_R r)[2\eta_R^{-2}(\alpha_1 - \alpha_2^{-1})$$
$$+ \alpha_2^{-1}] - (2\gamma_5^E/\Delta)E, \tag{3.5}$$

$$d_s^{-1}H(r) + \sum_{\lambda_n>0} J_1(\lambda_n r)[R_1^+(\lambda_n)\hat{F}(\lambda_n) + R_2^+(\lambda_n)\hat{G}(\lambda_n) + R_1^-(\lambda_n)\hat{H}(\lambda_n) - R_2^-(\lambda_n)\hat{K}(\lambda_n)] = 0,$$

$$d_n^{-1}K(r) + 2[(\gamma_2^E/\Delta) + \gamma_2^F]\hat{G}(0) - 2[(\gamma_2^E/\Delta) - \gamma_2^F + (\pi^2/4M\omega^2)]\hat{K}(0)$$
$$+ \sum_{\lambda_n>0} J_0(\lambda_n r)[Z_1^+(\lambda_n)\hat{F}(\lambda_n) + Z_2^+(\lambda_n)\hat{G}(\lambda_n) + Z_1^-(\lambda_n)\hat{H}(\lambda_n) - Z_2^-(\lambda_n)\hat{K}(\lambda_n)]$$
$$= -(2\gamma_5^E/\Delta)E - (M\omega^2)^{-1}F_2^c.$$

We define

$$\mathscr{F}(\alpha, \beta) = \int_0^1 J_1(\alpha r)J_1(\beta r)r\, dr,\ \mathscr{G}(\alpha, \beta) = \int_0^1 J_0(\alpha r)J_0(\beta r)r\, dr, \tag{3.6}$$

and, since $J_1(\lambda_n) = 0$, for $\lambda_n, \lambda_m > 0$ we have

$$\mathscr{F}(\lambda_n, \lambda_m) = \mathscr{G}(\lambda_n, \lambda_m) = \begin{cases} J_0^2(\lambda_n)/2 & \lambda_n = \lambda_m \\ 0 & \lambda_n \neq \lambda_m \end{cases}. \tag{3.7}$$

We apply the Hankel transforms $\mathscr{H}_1$ [eqns $(3.5)_{1,3}$; $r \to \lambda_m > 0$] and $\mathscr{H}_0$ [eqns $(3.5)_{2,4}$; $r \to \lambda_m > 0$], and, using eqns $(2.28)_5$ and (3.7) we obtain for $\lambda_n > 0$

$$R_1^-(\lambda_n)\hat{F}(\lambda_n) + R_2^-(\lambda_n)\hat{G}(\lambda_n) + R_1^+(\lambda_n)\hat{H}(\lambda_n) - R_2^+(\lambda_n)\hat{K}(\lambda_n) = [2iu_0/J_0^2(\lambda_n)]\mathscr{F}(\omega s_R, \lambda_n),$$

$$Z_1^-(\lambda_n)\hat{F}(\lambda_n) + Z_2^-(\lambda_n)\hat{G}(\lambda_n) + Z_1^+(\lambda_n)\hat{H}(\lambda_n) - Z_2^+(\lambda_n)\hat{K}(\lambda_n)$$
$$= [4iu_0\sigma/\pi J_0^2(\lambda_n)][2\eta_R^{-2}(\alpha_1 - \alpha_2^{-1}) + \alpha_2^{-1}]\mathscr{G}(\omega s_R, \lambda_n), \tag{3.8}$$

$$R_1^+(\lambda_n)\hat{F}(\lambda_n) + R_2^+(\lambda_n)\hat{G}(\lambda_n) + \{[2/d_s J_0^2(\lambda_n)] + R_1^-(\lambda_n)\}\hat{H}(\lambda_n) - R_2^-(\lambda_n)\hat{K}(\lambda_n) = 0,$$

$$Z_1^+(\lambda_n)\hat{F}(\lambda_n) + Z_2^+(\lambda_n)\hat{G}(\lambda_n) + Z_1^-(\lambda_n)\hat{H}(\lambda_n) + \{[2/d_n J_0^2(\lambda_n)] - Z_2^-(\lambda_n)\}\hat{K}(\lambda_n) = 0.$$

Applying the Hankel transforms $\mathscr{H}_0$ [eqns $(3.5)_{2,4}$; $r \to 0$] and using eqn $(2.28)_7$, we obtain

$$[(\gamma_2^E/\Delta) - \gamma_2^F]\hat{G}(0) - [(\gamma_2^E/\Delta) + \gamma_2^F]\hat{K}(0) = (2/\pi)\sigma^{1/2} iu_0[2\eta_R^{-2}(\alpha_1 - \alpha_2^{-1})$$
$$+ \alpha_2^{-1}][J_1(\omega s_R)/\omega s_R] - (\gamma_5^E/\Delta)E, \tag{3.9}$$
$$[(\gamma_2^E/\Delta) + \gamma_2^F]\hat{\mathscr{G}}(0) + [d_n^{-1} - (\gamma_2^E/\Delta) + \gamma_2^F - (\pi^2/4M\omega^2)]\hat{K}(0) = -(\gamma_5^E/\Delta)E - (2M\omega^2)^{-1}F_2^c.$$

In making the assumption of prescribed crystal bottom surface displacements we have removed the elastic scattered wave problem from the transducer/test medium interaction problem. The resulting problem still couples the crystal extensional and flexural deformations, transducer voltage and current, and casing motion. From Section 2 we see that these unknowns are expressed in terms of the interface tractions only through their finite Hankel transforms $\hat{F}(\lambda_n)$, $\hat{G}(\lambda_n)$, $\hat{H}(\lambda_n)$, $\hat{K}(\lambda_n)$, $\lambda_n > 0$, $\hat{G}(0)$ and $\hat{K}(0)$, and to obtain them we need not solve the coupled integral equations problem, eqns $(2.41)_{3,4}$ and $(3.3)_{1,3}$, for $F(r)$, $G(r)$, $H(r)$ and $K(r)$, $0 < r < 1$. We need only to solve the algebraic system of eqns (3.8) and (3.9) for $\hat{F}(\lambda_n)$, $\hat{G}(\lambda_n)$, $\hat{H}(\lambda_n)$, $\hat{K}(\lambda_n)$, $\lambda_n > 0$, $\hat{G}(0)$ and $\hat{K}(0)$.

We note that this algebraic problem decouples for each $\lambda_n \geq 0$. In particular, we solve eqn (3.9) to obtain

$$\begin{aligned} \hat{G}(0) = \{&[d_n^{-1} - (\gamma_2^E/\Delta) + \gamma_2^F - (\pi^2/4M\omega^2)](2/\pi)\sigma^{1/2} i u_0 [2\eta_R^{-2}(\alpha_1 - \alpha_2^{-1}) + \alpha_2^{-1}] \\ &\times [J_1(\omega s_R)/\omega s_R] + (\gamma_5^E/\Delta)[(\pi^2/4M\omega^2) - d_n^{-1} - 2\gamma_2^F]E - [(\gamma_2^E/\Delta) \\ &+ \gamma_2^F](2M\omega^2)^{-1} F_2^c\} D^{-1}, \\ \hat{K}(0) = \{&-[(\gamma_2^E/\Delta) + \gamma_2^F](2/\pi)\sigma^{1/2} i u_0 [2\eta_R^{-2}(\alpha_1 - \alpha_2^{-1}) + \alpha_2^{-1}][J_1(\omega s_R)/\omega s_R] \\ &+ (2\gamma_2^F \gamma_5^E/\Delta)E - [(\gamma_2^E/\Delta) - \gamma_2^F](2M\omega^2)^{-1} F_2^c\} D^{-1}, \end{aligned} \tag{3.10}$$

with

$$D = 4\gamma_2^E \gamma_2^F \Delta^{-1} + [(\gamma_2^E/\Delta) - \gamma_2^F][d_n^{-1} - (\pi^2/4M\omega^2)]. \tag{3.11}$$

From eqns (2.33), (2.34), (2.37) and (3.10) we obtain the desired transducer output voltage and current as

$$\begin{aligned} B = {}& 2de_{22}(\Delta D)^{-1}\{d_n^{-1} - \omega^{-2}[(\pi^2/4M) + 1]\}[2\eta_R^{-2}(\alpha_1 - \alpha_2^{-1}) \\ &+ \alpha_2^{-1}]\sigma^{1/2} i u_0 [J_1(\omega s_R)/\omega s_R] + (\pi\Delta)^{-1}(-\omega^2 + c_{22} \\ &+ 4\alpha e_{22}^2 d(\pi\Delta D)^{-1}\{d_n^{-1} - \omega^{-2}[(\pi^2/4M) + 2]\})E + d\pi e_{22}(2\Delta M\omega^4)^{-1} F_2^c, \\ I = {}& -(i\omega/d)(2B + E), \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} \Delta &= d(\omega^2 - c_{22} - 4\alpha e_{22}^2 \pi^{-1}) + 2(\omega^2 - c_{22})\pi^{-1}, \\ D &= [(d + 2\pi^{-1})\Delta^{-1} + (2\omega^2)^{-1}][d_n^{-1} - (\pi^2/4M\omega^2)] - 2(d + 2\pi^{-1})(\Delta\omega^2)^{-1}, \\ d &= -\omega^2 L - i\omega R + C^{-1}, d_n = -\omega^2 m_n - i\omega c_n + k_n. \end{aligned} \tag{3.13}$$

Hence, in eqns (3.12) and (3.13) we have expressed the transducer voltage and current amplitudes explicitly in terms of u_0, E, F_2^c, ω and the various material parameters.

Equation (3.8) can be solved separately for $\hat{F}(\lambda_n)$, $\hat{G}(\lambda_n)$, $\hat{H}(\lambda_n)$ and $\hat{K}(\lambda_n)$, for each $\lambda_n > 0$, in terms of u_0, ω and the material parameters. With these solutions and eqn (3.10), we can also express the torsionless axisymmetric crystal deformations $u_r^{(0)}$, $u_r^{(1)}$, $u_z^{(0)}$, $u_z^{(1)}$ and casing motion u_2^{CG}, in terms of u_0, E, F_2^c, ω and the material parameters, from eqns (2.21), (2.31), (2.34), (2.36) and (2.37).

Prescribed tractions on the crystal bottom surface

By specifying the torsionless axisymmetric traction components on the crystal bottom surface we simplify the problem through the elimination of $F(r)$ and $G(r)$ as unknowns, and hence we halve the number of integral equations to be solved. This approximation also allows a simple algebraic solution of the steady vibration interaction problem, in the manner of the previous approximation.

We restrict our attention to transducer/test medium systems in which the transducer is operating in the receiving mode, i.e. E and F_2^c are specified to be zero, so that the crystal/structure interface tractions are entirely due to the incident wave. Recall from eqn (2.8) that the incident wave is a straight crested surface wave travelling in the x_1' direction.

We assume the interface traction field is also a straight crested wave travelling in the x_1' direction, with the same speed and wave length as the incident wave. Hence, we assume

$$T_1(x_1, -b, x_3, t) = \text{Re}\{T_1(u_0)\,e^{-i\omega(t-s_R x_1)}\},\ T_3(x_1, -b, x_3, t) = 0,$$
$$T_2(x_1, -b, x_3, t) = \text{Re}\{T_2(u_0)\,e^{-i\omega(t-s_R x_1)}\}, \tag{3.14}$$

where the functions $T_1(u_0)$ and $T_2(u_0)$ must be specified. From eqn (2.4) we see that, for $0 < r < a$,

$$\mathring{T}_r^-(r) = iT_1(u_0)J_1(\omega s_R r),\ \mathring{T}_z^-(r) = T_2(u_0)J_0(\omega s_R r). \tag{3.15}$$

In nondimensional form with tildes deleted we therefore have, for $0 < r < 1$,

$$F(r) = iT_1(u_0)J_1(\omega s_R r),\ G(r) = T_2(u_0)J_0(\omega s_R r). \tag{3.16}$$

Equations $(3.16)_{1,2}$ replace eqns $(2.22)_{1-4}$ and also eqns $(2.41)_{1,2}$. The interface traction components $H(r)$ and $K(r)$ are still unknown and may be determined from the remaining integral equations, eqns $(2.41)_{3,4}$, with $F(r)$ and $G(r)$ specified by eqn (3.16).

In a manner similar to that used for the previous approximation, we may transform these integral equations and can then solve algebraically for $\hat{H}(\lambda_n)$, $\hat{K}(\lambda_n)$, $\lambda_n > 0$, and $\hat{K}(0)$. This allows an explicit representation of B, I, $u_r^{(0)}$, $u_r^{(1)}$, $u_z^{(0)}$, $u_z^{(1)}$ and u_2^{CG} in terms of $T_1(u_0)$, $T_2(u_0)$, ω and the various material parameters (see [12]). In particular, we obtain for the transducer output voltage

$$B = d\pi e_{22}\Delta^{-1}T_2(u_0)(1 + [(2\omega^2)^{-1} - (d + 2\pi^{-1})\Delta^{-1}]\{(d + 2\pi^{-1})\Delta^{-1} - d_n^{-1} \tag{3.17}$$
$$+ (2\omega^2)^{-1}[1 + (\pi^2/2M)]\}^{-1})[J_1(\omega s_R)/\omega s_R],$$

with

$$\Delta = d(\omega^2 - c_{22} - 4\alpha e_{22}^2\pi^{-1}) + 2(\omega^2 - c_{22})\pi^{-1},$$
$$d = -\omega^2 L - i\omega R + C^{-1},\ d_n = -\omega^2 m_n - i\omega c_n + k_n. \tag{3.18}$$

And, since $F(r)$ and $G(r)$ are specified, we may also give the scattered wave displacement in the test medium.

Transducer output for incident surface wave only

For the case of an incident surface wave with amplitude u_0, and in the absence of the forcing terms E and F_2^c, the approximations studied in this section lead to transducer output voltage formulas of similar form. Equation (3.12), obtained from the prescribed displacement assumption, can be written as

$$B(\omega) = f_d(\omega)u_0 J_1(\omega s_R), \tag{3.19}$$

whereas eqn (3.17), obtained from the prescribed traction assumption, also has the form

$$B(\omega) = f_t(\omega)u_0 J_1(\omega s_R), \tag{3.20}$$

provided we assume $T_2(u_0)$ is proportional to u_0.

The functions $f_d(\omega)$ and $f_t(\omega)$ are rational functions of ω, which also depend on several physical parameters. Some of these parameters, such as the viscoelastic foundation parameters m_n, c_n and k_n, are not known for a given transducer, and they would have to be determined by appropriate experiments before eqns (3.19) or (3.20) can be numerically evaluated.

As expected $B(\omega)$ in eqns (3.19) and (3.20) depends linearly on u_0. In addition, these expressions predict the same frequencies at which $B(\omega)$ vanishes. These null frequencies for $B(\omega)$ would be expected on the basis of the commonly observed phenomenon often referred to as "phase cancellation". Equations (3.19) and (3.20) predict

$$B(\omega_n) = 0, \tag{3.21}$$

where

$$J_1(\omega_n s_R) = 0, n = 1, 2, \ldots \tag{3.22}$$

or, from eqn (2.24),

$$\omega_n s_R = \lambda_n, \lambda_n > 0. \tag{3.23}$$

Taking into account the nondimensionalization, eqn (2.11), the null frequencies in dimensional form are

$$\omega_n = \lambda_n / s_R a, \lambda_n > 0. \tag{3.24}$$

In terms of the incident Rayleigh wavelength Λ_R this result becomes

$$\Lambda_R^{(n)} = 2\pi a / \lambda_n, n = 1, 2, \ldots \tag{3.25}$$

Since $\lambda_1 \simeq 3.831$, $\lambda_2 \simeq 7.016$, etc. it follows that

$$\Lambda_R^{(1)} \simeq 1.640a, \Lambda_R^{(2)} \simeq 0.8956a, \ldots \tag{3.26}$$

Therefore the first null in B occurs at an incident wavelength somewhat less than $2a$.

4. SUMMARY AND CONCLUSIONS

In this paper we have analyzed the transducer/test medium configuration of a piezoelectric transducer, typical of those commercially available, coupled viscously through its bottom surface to the test medium surface. The transducer was considered in both its sending and receiving modes of operation (for the sending transducer we did not specify the voltage across the piezoelectric crystal electrodes, but rather the voltage of a source in the electric circuit connecting the electrodes).

An important result from [12] is the demonstration that the voltage across the transducer crystal and current in the transducer circuit couple only with the torsionless axisymmetric parts of the transducer/test medium configuration's deformation and tractions. This is a result of the axisymmetric nature of our models for the transducer, test medium and couplant, as well as the presence of the face electrodes. Hence, the receiving transducer will only detect the torsionless axisymmetric component of its incident forcing, and the sending transducer will generate a torsionless axisymmetric signal in the test medium. Mathematically, this result leads to a significant simplification of the analysis as a general nonaxisymmetric problem is reduced to a torsionless axisymmetric one.

The steady vibration problem considered in this paper has the transducer forced by an incident surface wave in the test medium, a voltage source in the transducer circuit and tractions acting on the transducer casing. We reduced this problem to a set of coupled integral equations. In Section 3 we made assumptions on the nature of the transducer bottom surface/test medium surface interaction and were able to obtain the transducer voltage and current in terms of the forcing amplitudes, material parameters, and frequency of vibration, without solving the integral equations (a future task is to more carefully study the dependence of B and I on these quantities). In fact, for the assumptions of prescribed tractions on the transducer bottom surface we showed that the problem can be entirely solved (i.e. we may obtain the transducer voltage and current, and the torsionless axisymmetric parts of the transducer and test medium deformations in terms of the forcing amplitudes, material parameters, and frequency) algebraically, without solving any integral equations.

The conditions of phase cancellation in a circular transducer for straight crested incident surface waves was obtained for both approximations and is given by eqn (3.26).

In future work this condition will be compared with experiments and the more general results in eqns (3.12) and (3.17) will be computed for various assumptions on the parameters. These results will also be compared with experiments.

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